

# Two-Component Coupled KdV Equations and its Connection with the Generalized Harry Dym Equations

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## Abstract

It is shown that, three different Lax operators in the Dym hierarchy, produce three generalized coupled Harry Dym equations. These equations transform, via the reciprocal link, to the coupled two-component KdV system. The first equation gives us known integrable two-component KdV system while the second reduces to the known symmetrical two-component KdV equation. The last one reduces to the Driinfeld-Sokolov equation. This approach gives us new Lax representation for these equations.

# 1 Introduction

There are many different methods of the classification of integrable equations. The most popular is the utilizations of the conservation laws and the generalized symmetry. These methods led to the discovery of many new integrable systems [1, 2], both S-integrable and C-integrable in Calogero's terminology [3].

On the other side these methods have been used as well as to the classifications of the two-component coupled KdV type equations. For example Foursov in 2003 [4, 5] tested the integrability considering the following system of equations

$$u_t = F(u, v), \quad v_t = G(u, v) \quad (1)$$

where  $F(u, v) = F(u, v, u_x, v_x, u_{xx} \dots)$  denotes a differential polynomial function of  $u$  and  $v$ .

As the result Foursov presented five non-symmetrical [4] and 12 symmetrical [5] two-component coupled KdV systems which possesses several higher-order symmetries and conserved quantities. The symmetrical system is such where  $G(u, v) = F(v, u)$ .

The first three non-symmetrical systems in this classification are known to be integrable equations

$$u_t = u_{xxx} + 6uu_x - 12vv_x, \quad v_t = -2v_{xxx} - 6uv_x \quad (2)$$

$$u_t = u_{xxx} + 3uu_x + 3vv_x, \quad v_t = u_xv + uv_x \quad (3)$$

$$u_t = u_{xxx} + 2vu_x + uv_x, \quad v_t = uu_x \quad (4)$$

The first equation is the Hirota - Satsuma system [6], second is the Ito system [7], third is the rescaled Drinfeld - Sokolov equation [8].

The fourth system

$$\begin{aligned} u_t &= u_{xxx} + v_{xxx} + 2vu_x + 2uv_x \\ v_t &= v_{xxx} - 9uu_x + 6vu_x + 3uv_x + 2vv_x \end{aligned} \quad (5)$$

possesses several generalized symmetries and several higher order conserved densities and therefore Foursov conjectured that it is integrable and should possess infinitely many generalized symmetries.

The last system in this classification

$$\begin{aligned} u_t &= 4u_{xxx} + 3v_{xxx} + 4uu_x + vu_x + 2uv_x \\ v_t &= 3u_{xxx} + v_{xxx} - 4vu_x - 2uv_x - 2vv_x \end{aligned} \quad (6)$$

is integrable and possesses the Lax pair representations

$$L = (\partial_{xxx} + \frac{2}{3}u\partial_x + \frac{1}{3}u_x)(\partial_{xx} - \frac{1}{3}v), \quad \frac{\partial L}{\partial t} = 5[L, L_+^{3/5}] \quad (7)$$

and has been first considered many years ago by Drinfeld and Sokolov [8] and rediscovered by S. Sakovich [9]

In this paper we will discuss the problem how the coupled two-component KdV systems are connected with the generalized two component Harry Dym systems. It is

very well known that, Korteweg de Vries equation is connected with the Harry Dym equation [10, 11]. Strictly speaking the Harry Dym equation is reciprocally linked with the Korteweg de Vries equation. However the problem how it is possible to transform five of the mentioned two-component coupled KdV systems to some new generalization of the Harry Dym equations is an open problem. Such equivalence exists only for the Hirota-Satsuma equation [12, 13].

The one possible manner to find such connections is to apply the inverse reciprocal link to the coupled KdV systems. However in this procedure we have to assume the most general ansatz on the coupled modified KdV system. Next from these modified KdV functions we have to construct new functions, assuming once more the most general ansatz, in order to yield the generalized Harry Dym equation. This approach leads us to the algebraic system which is impossible to solve.

The next manner is to first find some generalization of the coupled Harry Dym equations and next construct the reciprocal link. If we progress it we expect to obtain the coupled KdV type systems. At the moment we know two different two-component generalizations of the Harry Dym equation. The first considered in [12, 13] is connected with the Hirota-Satsuma equation. The second, defined in [14], does not lead us to the coupled two component KdV system as we checked using the reciprocal link.

In this paper we use the second manner and present three Lax representations which produce new generalizations of the Harry Dym equation. Applying the reciprocal link to these equations we obtained the equations 6, the Drinfeld-Sokolov and one known symmetrical coupled KdV equation listed in [5].

The idea of the construction of the Lax representation for the generalized Harry Dym system follows from the observation that the Lax operator 7 which, generates the system 6, is exactly the product of two operators. The first is the Lax operator, which produces the Kupershmidt equation, while the second is the KdV Lax operator. It is known that an analogon of the Lax operator for the Kupershmidt equation, in the Dym hierarchy, is third order operator while an analogon of the Lax operator for the KdV equation is the Harry Dym Lax operator. If we consider the product of these two operators we expect to obtain some new generalization of the Harry Dym equation and in the next, by application of the reciprocal link, new interacted two-component KdV equations.

In principle we should consider three different operators because the Harry Dym equation possesses two different Lax operators the standard [15] and recently discovered the nonstandard [16]. We show that the Lax operator constructed as the product of third order and standard Harry Dym operators leads us to the equation 6. The operator constructed as the product of third order operator and non-standard Harry operator gives us known symmetrical two-component KdV equation. The Lax operator constructed as the product of standard and nonstandard Lax operator of Harry Dym leads us to the Drinfeld-Sokolov equation.

The paper is organized as follows. In the second section we present three Lax operators which generate three generalizations of the Harry Dym equation. The third section describes the reciprocal link from generalized Harry Dym equations to the coupled two-component KdV equations. In the fourth section we compare the Lax representation obtained using our approach with the known Lax representation for the coupled two-component KdV equations.

integrability properties of new coupled KdV equations. The last section contains concluding remarks.

## 2 New generalization of the Harry Dym equation

We meet three equivalent expression on the Harry Dym (HD) equation in the literature [11, 18]

$$w_t = (w^{-1/2})_{xxx}, \quad u_t = ((u_{xx})^{-1/2})_x, \quad v_t = v^3 v_{xxx} \quad (8)$$

where  $v = -2^{1/3}w^{-1/2}$  and  $u_{xx} = w$  respectively.

This equation have been discovered by H. Dym and M. Kruskal in 1975 [17] and share many of the properties typical of the soliton equations as for example it has a Bi-Hamiltonian structure

$$w_t = \mathcal{D}_1 \frac{\delta H_1}{\delta w} = \mathcal{D}_2 \frac{\delta H_2}{\delta w} \quad (9)$$

where

$$\begin{aligned} \mathcal{D}_1 &= \partial^3, & \mathcal{D}_2 &= \partial w + w \partial \\ H_1 &= 2 \int dx w^{-1/2}, & H_2 &= 8 \int dx w^{-5/2} w_x^2 \end{aligned} \quad (10)$$

and an infinite number of conservation laws and infinitely many symmetries [18]. This equation is connected with the Korteweg de Vries (KdV) equation via the reciprocal transformation [11, 10].

The HD equation follows from the following Lax representation.

$$L_s = \frac{1}{w} \partial^2, \quad L_{s,t} = 2[L_s, (L_s^{3/2})_{\geq 2}] \quad (11)$$

where subscript  $\geq 2$  denotes the projection to the part with the powers greater or equal to  $\partial^2$ .

On the other side there exists the second Lax operator for the Harry Dym equation, the nonstandard one, recently discovered in [16],

$$L_{ns} = w^{-1/4} \partial^{-1} w^{-1/4} \partial^2, \quad L_{ns,t} = [L_{ns}, (L_{ns}^3)_{\geq 2}]/2 \quad (12)$$

Now taking into account that the system of equation 6 has the Lax representation 7 in which the Lax operator is factorized as a product of the Lax operator of the Kupershmidt equation and the Lax operator of the Korteweg de Vries equation let us consider four Lax operators  $L_s, L_{sn}, L_{HS}, L_{DS}$

$$\begin{aligned} L &= w^3 \partial_{xxx} + k_0 w^2 w_x \partial_x \\ L_s &= L v^2 \partial^2, & L_{SD} &= L v \partial^{-1} v \partial^2 \\ L_{HS} &= w \partial^2 u \partial^2, & L_{DS} &= w^{-1/2} \partial^{-1} w^{-1/2} \partial^2 \frac{1}{u} \partial^2 \end{aligned} \quad (13)$$

The operator  $L_{HS}$  has been considered in [12] and leads us to the Hirota-Satsuma equation. From that reason we will not consider it in this paper.

Let us mention that the operator  $L$  generates by

$$L_t = [L_{\geq 2}^{5/3}, L] \quad (14)$$

the fifth order equation which, could be transformed by reciprocal link, to the Kupershmidt or Sawada-Kotera equation for  $k_0 = 3$  or  $k = \frac{3}{2}$  respectively.

The time evolution of the  $\hat{L}_s, \hat{L}_{DS}$  and  $L_{DS}$

$$L_{SD,t} = [(L_{SD}^{3/5})_{\geq 2}, L_{SD}], \quad L_{s,t} = [(L_s^{3/5})_{\geq 2}, L_s], \quad L_{DS,t} = [(L_{DS}^3)_{\geq 2}, L_{DS}] \quad (15)$$

produces the consistent solution only for  $k_0 = \frac{3}{2}$

For  $L_{SD}$  operator we obtained

$$w_t = \frac{2}{3}w^{5/2} \left( w^{3/2} (v^{6/5} w^{-6/5})_x \right)_{xx}, \quad v_t = \frac{1}{4}v^3 (w^{9/5} v^{-4/5})_{xxx} \quad (16)$$

while for  $L_s$

$$w_t = \frac{2}{3}w^{5/2} \left( w^{3/2} (v^{3/2} w^{-3/4})_x \right)_{xx}, \quad v_t = v^3 (w^{3/4} v_{xx} v^{-3/4})_x \quad (17)$$

For the  $L_{DS}$  operator we obtained

$$u_t = -\frac{1}{2} \left( \frac{1}{w} \right)_{xxx}, \quad w_t = -2 \left( \frac{\sqrt{w}}{u} \left( \frac{1}{\sqrt{w}} \right)_{xx} \right)_x \quad (18)$$

### 3 Reciprocal link

Introducing the parametrization of the function  $w, v$  for the  $L_{SD}$  operator as

$$w = ae^b, \quad v = ae^{-3b/2} \quad (19)$$

and for the  $L_s$  operator as

$$w = ae^b, \quad v = \sqrt{a}e^{-3b/2} \quad (20)$$

and for the  $L_{DS}$  operator as

$$u = \frac{1}{a}e^b, \quad w = \frac{1}{a^2}e^{-b} \quad (21)$$

we obtained that the  $L_{SD}$  operator generates

$$\begin{aligned} a_t &= \frac{1}{10}(-a_{xxx}a^3 + 9b_{xxx}a^3 + 27b_{xx}(a_xa^3 - 3b_xa^3) - \\ &\quad 81b_x^2a_xa^3 - 9b_x(a_{xx}a^3 - a_x^2a^2)), \\ b_t &= \frac{1}{10}(a_{xxx}a^2 + 11b_{xxx}a^3 + b_{xx}(33a_xa^2 - 9b_xa^3) + \\ &\quad 45b_x^3a^3 - 9b_x^2a_xa^2 + b_x(21a_{xx}a^2 + 6a_x^2a)) \end{aligned} \quad (22)$$

while the  $L_s$  operator gives us

$$\begin{aligned} a_t &= \frac{1}{4}(a_{xxx}a^3 - 9b_{xxx}a^4 - 27b_{xx}(a_xa^3 - b_xa^4) + 27b_x^2a_xa^3 - \\ &\quad 9b_x(a_{xx}a^3 + a_x^2a^2)) \\ b_t &= \frac{1}{4}(-a_{xxx}a^2 + b_{xxx}a^3 + 3b_{xx}(a_xa^2 + 3b_xa^3) - 18b_x^3a^3 + \\ &\quad 9b_x^2a_xa^2 - 3b_x(a_{xx}a^2 - a_x^2a)) \end{aligned} \quad (23)$$

and for the  $L_{DS}$  operator

$$\begin{aligned} a_t &= \frac{a^2}{2}(4a_{xx}a + b_x^2a^2 + 2(b_xa^2)_x)_x \\ b_t &= \frac{1}{2}(2a_x^3 - b_x^3a^3 - 2b_xa^2(b_xa_x + a_{xx}) - (2a_x^2a + b_x^2a^3 + 2b_xa_xa^2)_x) \end{aligned} \quad (24)$$

We use the reciprocal transformation where now  $x, a$  and  $b$  are defined as

$$x = p(y, t), \quad a(x, t) = p(y, t)_y, \quad b(x, t) = q(y, t) \quad (25)$$

in order to find the time evolution of  $p, q$ .

For  $L_{SD}$  case we obtained

$$\begin{aligned} p_t &= \frac{1}{20}(-2p_{yyy} + 3\frac{p_{yy}^2}{p_y} + p_y(18q_{yy} - 81q_y^2)) \\ q_t &= \frac{1}{20}\left(2\frac{p_{4y}}{p_y} - p_{yy}\left(8\frac{p_{yyy}}{p_y^2} - 6\frac{p_{yy}^2}{p_y}\right) + 22q_{yyy} + q_y(9q_y + 18\frac{p_{yyy}}{p_y} - 27\frac{p_{yy}^2}{p_y^2})\right) \end{aligned} \quad (26)$$

while for  $L_s$

$$\begin{aligned} p_t &= \frac{1}{8}(2p_{yyy} - 3\frac{p_{yy}^2}{p_y} - p_y(18q_{yy} - 27q_y^2)) \\ q_t &= \frac{1}{8}(-2\frac{p_{4y}}{p_y} + p_{yy}\left(8\frac{p_{yyy}}{p_y^2} - 6\frac{p_{yy}^2}{p_y}\right) + 2q_{yyy} - q_y(9q_y + 6\frac{p_{yyy}}{p_y} - 9\frac{p_{yy}^2}{p_y^2})) \end{aligned} \quad (27)$$

and for the  $L_{DS}$  operator

$$\begin{aligned} p_t &= \frac{1}{4}(4p_{yyy} - 4\frac{p_{yy}^2}{p_y} + 2q_{yy}p_y + q_y^2p_y + 2q_y p_{yy}) \\ q_t &= \frac{1}{4}\left(-8p_{yyy}p_{yy}p_y^{-2} + 8p_{yy}^3p_y^{-3} - 2q_{yy}q_y - q_y^3 - 4p_y^{-1}(q_{yy}p_{yy} + q_y^2p_{yy} + q_y p_{yyy})\right) \end{aligned} \quad (28)$$

To verify this one can use the identities

$$\partial_x = \frac{1}{a}\partial_y, \quad a_t = p_{y,t} - \frac{p_{yy}p_t}{p_y}, \quad b_t = q_t - \frac{q_y p_t}{p_y} \quad (29)$$

Next we apply the transformation

$$f = \frac{p_{yy}}{p_y}, \quad g = q_y \quad (30)$$

from which we conclude that for the  $L_{SD}$  operator we obtained

$$\begin{aligned} f_t &= \frac{1}{20} \left( -2f_{yy} + f^3 + 18g_{yy} + 18g_y f - 162gg_y - 81g^2 f \right)_y \\ g_t &= \frac{1}{20} \left( 2f_{yy} - 2f_y f + 22g_{yy} + 9g^3 + 18gf_y - 9gf^2 \right)_y \end{aligned} \quad (31)$$

while for the  $L_s$  operator

$$\begin{aligned} f_t &= \frac{1}{8} (2f_{yy} - f^3 - 18g^2 - 18g_y f + 54g_y g + 27g^2 f)_y \\ g_t &= \frac{1}{8} (-2f_{yy} + f_y f + 2g_{yy} - 9g^3 - 6gf_y + 3gf^2)_y \end{aligned} \quad (32)$$

and for the  $L_{DS}$  operator

$$\begin{aligned} f_t &= \frac{1}{4} \left( (4f_y + 2g_y + 2f^2 + g^2 + 2gf)_y + 2g_y f + g^2 f + 2gf^2 \right)_y \\ g_t &= -\frac{1}{4} \left( (4f^2 + g^2 + 4gf_y)_y + g^3 + 4g^2 f + 4gf^2 \right)_y \end{aligned} \quad (33)$$

If we apply the Miura-type transformation for the  $L_{SD}$  case

$$\begin{aligned} u &= \frac{1}{8} (8f_y + 3f^2 + 27g^2 - 6fg) \\ v &= \frac{1}{24} (24g_y - f^2 - 9g^2 + 18fg) \end{aligned} \quad (34)$$

then the system of equation 33 reduces to

$$\begin{aligned} u_t &= \frac{1}{20} (-2u_{yyy} + 18v_{yyy} + 9uu_y - 9uv_y - 405vv_y - 27vu_y) \\ v_t &= \frac{1}{20} (2u_{yyy} + 22v_{yyy} - 5uu_y - 27uv_y + 9vv_y - 9vu_y) \end{aligned} \quad (35)$$

If we apply the linear transformation of the variables  $u, v$  and apply the scale of the time

$$t \Rightarrow \frac{t}{4}, \quad u \Rightarrow \frac{v - u}{3}, \quad v \Rightarrow -\frac{u + v}{9} \quad (36)$$

then the system 35 reduces to the Drinfeld-Sokolov equation.

Applying different Miura-type transformation for the  $L_s$  case

$$u = \frac{1}{2} (2f_y - f^2), \quad v = \frac{1}{2} (2g_y - 3g^2) \quad (37)$$

we obtained

$$\begin{aligned} u_t &= \frac{1}{4} (u_{yyy} + 3uu_y - 9v_{yyy} - 18uv_y - 9vu_y) \\ v_t &= \frac{1}{4} (-u_{yyy} + v_{yyy} - 3uv_y - 6vu_y + 9vv_y) \end{aligned} \quad (38)$$

Applying the scaling  $u \Rightarrow \frac{2}{3}u, v \Rightarrow \frac{2}{9}v$  the system of equation 38 reduces to the symmetrical form

$$u_t = -2\left(-\frac{1}{2}u_{yyy} - uu_y + \frac{3}{2}v_{yyy} + 2v_yu + vu_y\right). \quad (39)$$

It is one of the rescaled symmetrical coupled two-component KdV system considered by Foursov [5]<sup>1</sup>.

If we apply the Miura-type transformation

$$s = f_y - \frac{1}{2}f^2, \quad z = g_y + f^2 + \frac{1}{2}g^2 + fg \quad (40)$$

for the  $L_{DS}$  case, then the system of equation 33 reduces to

$$s_t = \frac{1}{2}\left((2s_{yy} + z_{yy} + 3s^2 + sz)_x + z_xs\right)_y \quad (41)$$

$$z_t = -\frac{3}{4}(4s^2 + 2zs + z^2)_y. \quad (42)$$

If we further shift the function  $s \Rightarrow s - z/2$  we obtain

$$s_t = \frac{1}{2}(2s_{xxx} - z_xs - 2zs_x) \quad (43)$$

$$z_t = -6s_y s. \quad (44)$$

It is exactly the first Drinfeld - Sokolov equation DS1 if we apply the scaling  $s \Rightarrow s/2, z \Rightarrow -z, \partial_t \Rightarrow \partial_t/2$ .

## 4 The Lax representations of the coupled two-component KdV type systems

The known Lax representation of the Drienfeld-Sokolov [8] is

$$L = (\partial^3 + (s+z)\partial + \frac{1}{2}(s+z)_x)(\partial^3 + (s-z)\partial + \frac{1}{2}(s-z)_x), \quad L_t = [L_{\geq 0}^{1/2}, L] \quad (45)$$

while for the system 6 is 7.

We show that our approach produces quite different Lax representation for these equations.

First let us consider the  $L_{DS}$  operator for which we apply the gauge transformation

$$\hat{L}_{DS} = e^{-b/2} L_{DS} e^{b/2} \quad (46)$$

and next the reciprocal transformation and once more the gauge transformation

$$\hat{L}_{DS} \Rightarrow \frac{1}{p_y} \hat{L}_{DS} p_y \quad (47)$$

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<sup>1</sup>the equation 4.9 in [4] where  $\alpha = 1$ .



Finally if we apply the Miura-type transformation 40 we obtain

$$\begin{aligned}\hat{L}_{DS} = & \partial^3 + (3s + 2z)\partial + \partial^{-1}\left(s_{yy} + \frac{1}{2}z_{yy} - \frac{1}{4}z^2 - \frac{1}{4}zs\right) - \\ & \frac{3}{2}(f + g)^2\partial + \partial^{-1}\left((f + g)\left(\frac{1}{2}\left(s + \frac{1}{2}z\right)(f + g) - s_y - \frac{1}{2}z_y\right)\right)\end{aligned}\quad (48)$$

Strictly speaking we obtained the Lax operator which generates the time evolution of the function  $f, g$  and implicitly the Drinfeld-Sokolov equation. Interestingly this operator produces also the conserved quantities for this equation. It can be easily seen for the first quantities

$$H_0 = \text{Res}(L) = \int dy \, s^2 + \frac{1}{4}z^2 + zs. \quad (49)$$

For the  $L_s$  operator making the similar transformations we obtained

$$\begin{aligned}\hat{L}_s = & \partial_{4x} + (u + 3v)\partial_{xx} + \frac{3}{2}(u + 3v)_x\partial + \frac{1}{2}(u + 3v)_{xx} + \\ & \frac{1}{4}(u - 3v)^2 + \frac{1}{4}(u - 3v)_x\partial^{-1}(u - 3v)\end{aligned}$$

For the  $L_{DS}$  operator making the similar transformations we obtained

$$\begin{aligned}\hat{L}_{DS} = & e^{-\int dy((f+3g)/2)} L_{DS} e^{\int dy((f+3g)/2)} = (\partial_{yyy} - 3\partial_{yy}g + \frac{1}{4}\partial_y(2f_y - f^2 + 6g_y + 9g^2)) \\ & (\partial_{yy} + 3g\partial_y + \frac{1}{4}(2f_y - f^2 + 6g_y + 9g^2)).\end{aligned}\quad (50)$$

The time evolution of  $\hat{L}_{DS}$

$$\hat{L}_{DS,t} = [(\hat{L}_{DS}^{3/5})_+, \hat{L}_{DS}] \quad (51)$$

generates the coupled mKdV equation 31 and in the implicit form also the system of 6. However  $\hat{L}_s$  can not be rewritten, after applications of the Miura transformation 34, purely in terms of the local KdV variables  $u, v$  and its derivatives. As we checked the residuum formula of the  $\hat{L}$  generates the conserved quantity for the MKdV equation as well as for the coupled KdV system 35. Indeed the first two nontrivial conserved laws obtained from  $\hat{L}$  operator are

$$\begin{aligned}G_2 = & \text{res}(\hat{L}^{1/5}) = \int (f^2 + 9g^2)dy \\ G_4 = & \text{res}(\hat{L}^{3/5}) = \int (4f_{yy}f - f^4 - 72g^2f - 396g_{yy}g - 36g_yf^2 + \\ & 648g_ygf - 81g^4 + 162g^2f^2)dy\end{aligned}\quad (52)$$

Here the lower index in  $G, H$  denotes the KdV weight of the function e.g  $[u] = [v] = 2, [f] = [g] = 1, [\partial] = 1$ . In order to obtain the conserved quantities for the equation 6 from the  $\hat{L}_s$  operator let us apply the Miura transformation 40 and rewrite these quantities as

$$\begin{aligned}G_2 = & \int (f^2 + 9g^2 + 3f_y + 3g_y)dy = 3 \int (u + v)dy = H_2 \\ G_4 = & \int dy(u^2 - 99v^2 - 18uv).\end{aligned}\quad (53)$$

In the similar way it is possible to obtain the higher order conserved densities.

## 5 Concluding remarks

In this paper we presented three different generalizations of the Harry Dym equation. These equations have been obtained using different Lax operators in the Harry Dym hierarchy. Using the reciprocal link we showed that our equations are reduced to the coupled two-component KdV equations. However how remaining coupled two-component KdV equations listed in [4, 5] are connected with the generalized Harry Dym equations is still an open problem.

## References

- [1] A. Mikhailov, A. Shabat, V. Sokolov *The symmetry approach to classification of integrable equations* in *What is Integrability* edited by V. Zakharov (Springer-Verlag 1991) 115-184.
- [2] A. Mikhailov, A. Shabat, R. Yamilov *The symmetry approach to the classification of nonlinear equations. Complete lists of integrable systems* Russ. Math. Surveys **42** (1987) 115-184.
- [3] F. Calogero *Why are certain nonlinear PDEs both widely applicable and integrable* in *What is Integrability* edited by V. Zakharov (Springer-Verlag 1991) 1-62.
- [4] M. Foursov *Towards the complete classification of homogenous two-component integrable equations* J. Math. Phys. **44** (2003) 3088-3096.
- [5] M. Foursov *On integrable coupled KdV-type systems* Inverse Problems **16** (2000), 259 - 274.
- [6] R. Hirota, J. Satsuma *Soliton solutions of a coupled Korteweg-de Vries equation* Phys. Lett **85A** 407-408.
- [7] M. Ito, *Symmetries and conservation laws of a coupled nonlinear wave equation*, Phys. Lett. A, 1982, V.91, 335-338.
- [8] V. Drinfeld, Sokolov V, *New evolutionary equations possessing an  $(L, A)$ -pair*, Trudy Sem. S.L. Soboleva (1981), no.2 5-9 (in Russian).
- [9] S. Sakovich *Coupled KdV Equations of Hirota-Satsuma Type* Journal of Nonlinear Mathematical Physics 1999, V.6 N 3 255 -262, *ibid* Addendum to: *Coupled KdV Equations of Hirota-Satsuma Type* Journal of Nonlinear Mathematical Physics Volume 6, Number 2 (2001), 311-312.
- [10] N.H Ibragimov *Sur l'équivalence des équations d'évolution, qui admettent une algèbre de Lie-Bäcklund infinie* C. R. Acad. Sci., Paris (1981) 293 657-60.
- [11] W. Hereman, P. P. Banerjee and M. R. Chatterjee, *On the Nonlocal Equations and Nonlocal Charges Associated with the Harry Dym Hierarchy Korteweg-de Vries equation* J. Phys. A **22** (1989) 241.

- [12] Z. Popowicz *The Generalized Harry Dym Equation* Phys. Lett. **A** 317 260-264 (2003).
- [13] S. Sakovich *Transformation of a generalized HD equation into the Hirota-Satsuma system* Phys. Lett. A321 (2004) 253-254.
- [14] M. Antonowicz, A. Fordy *Coupled Harry Dym equations with multi-Hamiltonian structures* J. Phys. A: Math. Gen. **21** (1988) L269-L275.
- [15] B. Konopelchenko, W.Oevel; *An r-Matrix Approach to Nonstandard Classes of Integrable Equations* Publ. Rims Kyoto Unive. 29 (1993) 58.
- [16] K. Tian, Z. Popowicz, Q. Liu *A non-standard Lax formulation of the Harry Dym hierarchy and its supersymmetric extension* J. Phys A:Math.Theor. 45 (2012) 122001 (8pp).
- [17] M.D. Kruskal *Lecture Notes in Physics* vol.38, Springer Berlin 1975, p. 310.
- [18] J.C. Brunelli, G.A.T.F. da Costa *On the Nonlocal Equations and Nonlocal Charges Associated with the Harry Dym Hierarchy* J.Math. Phys. 43 (2002) 6116-6128.